Measurable and Unmeasurable in "Protective" Measurements

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Recently proposed idea of "protective" measurement of a quantum state is critically examined, and generalized. Earlier criticisms of the idea are discussed and their relevance to the proposal assessed. Several constraints on measuring apparatus required by "protective" measurements are discussed, with emphasis on how they may restrict their experimental feasibility. Though "protective" measurements result in an unchanged system state and a shift of the pointer proportional to the expectation value of the measured observable in the system state, the actual reading of the pointer position gives rise to several subtleties. We propose several schemes for reading pointer position, both when the apparatus is treated as a classical system as well as when its quantum aspects are taken into account, that address these issues. The tiny entanglement which is always present due to deviation from extreme adiabaticity in realistic situations is argued to be the weakest aspect of the proposal. Because of this, one can never perform a protective measurement on a single quantum system with absolute certainty. This clearly precludes an ontological status for the wave function. Several other conceptual issues are also discussed.

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I. INTRODUCTION

Quantum mechanics is a theory which has been tremendously successful in explaining how the physical world works, but its measurement aspects have been plagued with interpretational problems since its inception. The general credo is that the value of a real physical observable, described by a Hermitian operator, has meaning only when the system is in its eigenstate, i.e.,

$$A|a_i\rangle = a_i|a_i\rangle, \tag{1}$$

where a_i is the eigenvalue of A corresponding to the eigenstate $|a_i\rangle$. Furthermore, if the system is in a state $|n\rangle$ which is not an eigenstate of A, a measurement of A can yield as a result any of the eigenvalues of A while "collapsing" $|n\rangle$ to $|a_i\rangle$ at the same time. Thus the outcome of a single measurement on a single quantum system can not be assigned any significance. As a corollary, the state of a single quantum system can not also be attributed any objective significance. The statistical interpretation, originating in the early works of Einstein [1], can be considered the "optimal way out" for this strange aspect of quantum phenomena. According to this, if $|n\rangle$ has the (unique) expansion

$$|n\rangle = \sum_i c_i|a_i\rangle \tag{2}$$

the outcome of a large number of measurements of A on an ensemble of identically prepared states are a_i with probability $|c_i|^2$ and the "expectation value" of A in $|n\rangle$ is construed as the ensemble average $\sum_i |c_i|^2 a_i$. The eigenvalue condition (1) can be interpreted as a sort of consistency condition for this interpretation. Clearly any other state $|\tilde{n}\rangle = \sum \tilde{c}_i|a_i\rangle$ with $\tilde{c}_i = e^{i\phi_i}c_i$ will also yield an identical distribution of a_i as $|n\rangle$ in an ensemble measurement of A. To determine $|n\rangle$, therefore, many ensemble measurements have to be carried out with different observables. The number of such independent ensemble measurements needed to determine the original state is dictated by the "size" of the density matrix which is the number of independent parameters needed to specify the density matrix.

Apart from granting only an "epistemological" meaning to the quantum state (wave function), this interpretation leads to a notion of reality fundamentally different from that in classical mechanics. It also puts observation or measurements on a totally different footing than in classical mechanics (as John Wheeler has succinctly put it, "no phenomenon is a phenomenon until it is an observed phenomenon"). At the same time, the notion of "collapse" or the "projection postulate" as enunciated by von Neumann [2] leads to its own set of conceptual difficulties. As the density matrix of a pure state ($tr\rho = 1 = tr\rho^2$) turns into that of a mixed state ($tr\rho = 1, tr\rho^2 < 1$) after the ensemble measurement, something that can never be achieved through an unitary evolution, it appears as if new elements have to be introduced into the theoretical framework to accommodate the measurement process. This in a nutshell is the "measurement problem" of quantum theory. Proposals to "solve" this fantastic situation are even more fantastic like the Everett many worlds interpretation [3] or the GRW proposal [4]. As there are no feasible means of experimentally testing these at the moment, they remain as merely matters of individual taste.

For a single quantum state, the situation is even more complex. When the state is *a priori* unknown, measurement of any observable is generically not going to be an eigenstate measurement. Consequently, after the measurement, the state of the system will change in an uncontrollable manner. Any number of subsequent measurements are not going to give information about the original state i.e the average values of the outcome of repeated measurements have no bearing on the expec-

tation value of the observable in the original state (for an interesting twist to this see section IV-D). Of course, the expectation value of any observable A in an a priori known state $|n\rangle$ can always be calculated. In such a situation one can also come up with schemes to to perform a "measurement" of the expectation value as well as the associated variance by either using so called reversible measurements [5], or by avoiding entanglement. But one doesn't gain any new information about the system. Even the a priori known wavefunction is verified only in a statistical sense. In fact one is only performing an ensemble measurement in disguise. Thus neither the generic (as opposed to a priori unknown) state of a single quantum system nor expectation values of observables in it can be given any meaning. The standard lore, therefore, denies any "reality" or "ontological" meaning to the wave function.

Therefore, the recent proposal by Aharonov, Anandan and Vaidman (AAV) [6–9] of a scheme involving adiabatic measurements, which they have called "protective" measurements, wherein they have claimed the possibility of measurement of $\langle A\rangle$ in the state $|n\rangle$ of a single quantum system for any observable A, without disturbing $|n\rangle$, has indeed raised surprise and scepticism among many [10–17]. This proposal is remarkable from many points of view, all of a fundamental nature, and therefore deserves the most careful scrutiny. AAV claim to be able to measure $\langle A\rangle_n$ for any $A, |n\rangle$ whereas we saw that the standard lore does not allow it even if one is willing to uncontrollably disturb $|n\rangle$. Even more remarkably, they claim to be able to do so without disturbing the system at all. This allows for these protective measurements to be repeated with sufficiently many observables to completely determine the state modulo an overall phase. Here again, the number of different observables to be protectively measured in order to determine the state of the system is governed by the number of independent parameters in the density matrix. Thus their proposal, as stressed by them, allows for an "ontological" meaning to the wave function of a single system.

AAV have made many proposals to realize such protective measurements which can be broadly split into two categories: i) a Quantum Zeno type measurements made on an *a priori known* state of the single system and ii) an adiabatic measurement made on *an a priori unknown* state of the system which, however, is *known* to be a non-degenerate eigenstate of *an a priori unknown Hamiltonian*. Here we restrict our attention to only the second category which we feel is the more interesting one. A number of criticisms of this proposal have appeared subsequently [10–17]. In this paper we critically review and assess the original proposal as well as the criticisms. We also extend the scope and generality of both.

The paper is organised as follows: in section II we present the idea of protective measurements in a rigorous way, and then go on to generalize it. We also discuss a few examples which highlight some subtle points regarding the original AAV proposal. In section III, we critically analyze various criticisms of the original AAV proposal and assess their relevance to the issue. In section IV we discuss the very important issue of spreading of the pointer position and suggest some ways to circumvent the problem. In section V we make detailed remarks on the restrictions imposed on the measuring apparatus by protective measurements, and the feasibility of practical implementation of the idea. We also discuss the relevance of protective measurement to the issue of the "reality" of the wave-function. Finally, in section VI we summarise the main results of the present investigation.

A more rigorous derivation as well as generalisation of the original AAV proposal(sec II), a discussion of the relevance of the degeneracy of the total (system and apparatus) Hamiltonian with examples (sec IIB), a careful treatment of the effects of switching on/off of the apparatus-system interaction (sec IIC), an unambiguous rephrasing of the AAV spin-1/2 example (sec IIE) are new features of this paper designed to bring greater clarity to the discussion. Secs III-V are totally new contributions.

II. PROTECTIVE MEASUREMENT

Let us first consider a conventional measurement. Let Q_S be an operator, corresponding to the observable of the system we wish to measure, and let it interact with an appropriate apparatus(in what follows, we shall use the notion of an apparatus to indicate a quantum system to which full information about the system can be transferred) through an interaction

$$H_I = g(t)Q_A Q_S, \tag{3}$$

where Q_A is an observable of the apparatus, and $g(t)$ is the strength of the interaction normalized such that $\int dt g(t) = 1$. The interaction is nonzero only in the short interval $[0, \tau]$. Let the system be in an initial state $|\nu\rangle$ which is not necessarily an eigenstate of Q_S, and the apparatus be in a state $|\phi(r_0)\rangle$, which is a wave packet of eigenstates of the operator R_A conjugate to Q_A, centered at the eigenvalue r_0. The interaction H_I is of short duration, and assumed to be so strong that the effect of the free Hamiltonians of the apparatus and the system can be neglected. Then the combined wave function of the system and the apparatus at the end of the interaction can be written as

$$|\psi(\tau)\rangle = e^{-\frac{i}{\hbar}Q_A Q_S}|\nu\rangle|\phi(r_0)\rangle. \tag{4}$$

If we expand $|\nu\rangle$ in the eigenstates of Q_S, $|s_i\rangle$, we obtain

$$|\psi(\tau)\rangle = \sum_i e^{-\frac{i}{\hbar}Q_A s_i} c_i |s_i\rangle|\phi(r_0)\rangle, \tag{5}$$

where s_i are the eigenvalues of Q_S and c_i are the expansion coefficients. The exponential term shifts the center of the wave packet by s_i:

$$|\psi(\tau)\rangle = \sum_i c_i |s_i\rangle |\phi(r_0 + s_i)\rangle. \quad (6)$$

This is an entangled state, where the position of the wave packet gets correlated with the eigenstates $|s_i\rangle$. Detecting the center of the wave packet at $r_0 + s_i$ will throw the system into the eigenstate $|s_i\rangle$.

Protective measurements, on the other hand, make use of the opposite limit where the interaction of the system with the apparatus is *weak* and *adiabatic*. Here the system is assumed to be in a non-degenerate eigenstate of its Hamiltonian, and the interaction being weak and adiabatic, we cannot neglect the free Hamiltonians. Let the Hamiltonian of the combined system be

$$H(t) = H_A + H_S + g(t) Q_A Q_S, \quad (7)$$

where H_A and H_S are the Hamiltonians of the apparatus and the system, respectively. The coupling $g(t)$ acts for a long time T and goes to zero smoothly before and after the interaction. It is also normalized as $\int_0^T dt g(t) = 1$. Therefore, $g(t) \approx 1/T$ is small and constant for the most part. If $|t = 0\rangle$ is the state vector of the combined apparatus-system just before the measurement process begins, the state vector after T is given by

$$|t = T\rangle = \mathcal{T} e^{-\frac{i}{\hbar}\int_0^T H(\tau) d\tau} |t = 0\rangle, \quad (8)$$

where $\mathcal{T}$ is the time ordering operator. We divide the interval $[0, T]$ into N equal intervals ΔT, so that $\Delta T = T/N$, and because the full Hamiltonian commutes with itself at different times during $[0, T]$, we can write eqn(8) as

$$|t = T\rangle = \left(exp[-\frac{i\Delta T}{\hbar}(H_A + H_S + \frac{1}{T} Q_A Q_S)] \right)^N |t = 0\rangle. \quad (9)$$

Let us now examine the case when Q_A commutes with the free Hamiltonian of the apparatus, i.e., $[Q_A, H_A] = 0$, so that we can have eigenstates $|a_i\rangle$ such that $Q_A|a_i\rangle = a_i|a_i\rangle$ and $H_A|a_i\rangle = E_i^a|a_i\rangle$. Choudhury, Dasgupta and Datta [14] consider only two cases: one where $[Q_A, H_A] = 0$ and $[Q_S, H_S] = 0$, second where $[Q_A, H_A] \neq 0$ and $[Q_S, H_S] \neq 0$. Thus they put an additional restriction that Q_A and Q_S either commute or do not commute with the unperturbed Hamiltonian, together, and miss the important case where $[Q_A, H_A] = 0$ and $[Q_S, H_S] \neq 0$. Now $|a_i\rangle$ are also the exact eigenstates of the instantaneous Hamiltonian $H(t)$, in the apparatus subspace. So, the exact instantaneous eigenstates can be written in a factorized form $|a_i\rangle\overline{|\mu\rangle}$ where $\overline{|\mu\rangle}$ are system states which depend on the eigenvalue of Q_A, *i.e.*, they are the eigenstates of $\frac{1}{T} a_i Q_S + H_S$. Let us assume the initial state to be a direct product of a non-degenerate eigenstate of H_S, $|\nu\rangle$, and $|\phi(r_0)\rangle$:

$$|t = 0\rangle = |\nu\rangle |\phi(r_0)\rangle. \quad (10)$$

Introducing complete set of exact eigenstates in the above equation, the wave function at a time T can now be written as

$$|t = T\rangle = \sum_{i,\mu} e^{\frac{i}{\hbar} E(a_i,\mu) N \Delta T} |a_i\rangle \overline{|\mu\rangle\langle\mu|} |\nu\rangle\langle a_i| |\phi(r_0)\rangle, \quad (11)$$

where the exact instantaneous eigenvalues $E(a_i, \mu)$ can be written as

$$E(a_i, \mu) = E_i^a + \frac{1}{T}\overline{\langle\mu|} Q_S \overline{|\mu\rangle} a_i + \overline{\langle\mu|} H_S \overline{|\mu\rangle}. \quad (12)$$

Till here the treatment is exact, except for ignoring the switching on and switching off times to begin with. We justify ignoring these in sec IIC. It should be kept in mind that the expectation value $\langle Q_S\rangle_{\overline{\mu}}$ depends on the eigenvalue a_i of Q_A. The sum over μ in (11) makes it appear as if the state is entangled. But the important point to notice is that the basis $\overline{|\mu\rangle}$ can be made to be *arbitrarily* close to the original basis, as the interaction is assumed to be weak, so that $\overline{|\mu\rangle} = |\mu\rangle + \mathcal{O}(1/T) + ...$ In the large T limit, one can assume the states to be unperturbed, and retain only terms of $O(1/T)$ in the energy (this is necessary as $E(a_i, \mu)$ is multiplied by T in eqn(11)), which amounts to using first order perturbation theory. This yields eigenvalues of the form

$$E(a_i, \mu) = E_i^a + \frac{1}{T}\langle\mu|Q_S|\mu\rangle a_i + \langle\mu|H_S|\mu\rangle + O(1/T^2). \quad (13)$$

In addition to this, the sum over μ disappears and only the term where $\mu = \nu$ survives. Thus, we can write the apparatus part of the exponent again in the operator form

$$|t = T\rangle \approx e^{-\frac{i}{\hbar} H_A T - \frac{i}{\hbar} Q_A \langle Q_S\rangle_\nu - \frac{i}{\hbar}\langle H_S\rangle_\nu T} |\nu\rangle |\phi(r_0)\rangle. \quad (14)$$

Now, it is easy to see that the second term in the exponent will shift the center of the wave packet $|\phi(r_0)\rangle$ by an amount $\langle\nu|Q_S|\nu\rangle$:

$$|\psi(T)\rangle = e^{-\frac{i}{\hbar} H_A T - \frac{i}{\hbar}\nu T} |\nu\rangle |\phi(r_0 + \langle Q_S\rangle_\nu)\rangle. \quad (15)$$

This shows that at the end of the interaction, the center of the wave packet $|\phi(r_0)\rangle$ shifts by $\langle\nu|Q_S|\nu\rangle$.

The idea behind this approximation is that in $\overline{\langle\mu|}\nu\rangle$ only one term is large and close to unity, and rest of the terms are very small, of the order $1/T$. Making T very large, one can make the smaller terms arbitrarily close to zero. Thus, the state is effectively not entangled, and so the original wave function is not destroyed during the measurement. Looking at the position of the wave packet, one can determine the expectation value $\langle Q_S\rangle_\nu$. This, basically, is the essence of the argument for protective measurements, although it was not shown with this much rigour in the original proposal. Further, it has been asserted that one *needs* the condition $[Q_A, H_A] = 0$ to obtain a clean protective measurement [9]. In the following we will show that this condition is not really necessary for a protective measurement, and the idea can be made quite general.

A. The general case

We consider again the Hamiltonian in (7). As we are interested in examining the possibility of protective measurements in the most general context,

$$[H_A, Q_A] \neq 0 \qquad [H_S, Q_S] \neq 0. \tag{16}$$

T denotes the duration of the adiabatic measurement. If $|t = 0\rangle$ is the state vector just before the measurement process begins, the state vector after T is again given by (8). Here again, with $g(t) = 1/T$, the Hamiltonian is time-independent and no time-ordering is needed. In that case

$$|t = T\rangle = e^{iTH}|t = 0\rangle, \tag{17}$$

where

$$H = H_A + H_S + \frac{Q_A Q_S}{T}. \tag{18}$$

We start with an initial state satisfying the conditions laid down by [6–8]

$$|t = 0\rangle = |\nu\rangle|\phi\rangle, \tag{19}$$

where $|\nu\rangle$ is a non-degenerate eigenstate of H_S and $|\phi\rangle$ is a general state of the apparatus, not necessarily an eigenstate of H_A (which we shall denote generically by $|a\rangle$). Then

$$|t = T\rangle = e^{iHT}|\nu\rangle|\phi\rangle. \tag{20}$$

We further expand $|\phi\rangle$ in the basis $|a\rangle$ and write

$$|t = T\rangle = e^{iHT}\sum_b d_b|\nu\rangle|b\rangle. \tag{21}$$

Denoting the exact eigenstates of H by $|\Psi_{\mu,a}\rangle$ and the corresponding eigenvalues by $E(\mu, a)$, we have

$$|t = T\rangle = \sum_b d_b \sum_{\mu,a} e^{iE(\mu,a)T}\langle\Psi_{\mu,a}|\nu, b\rangle|\Psi_{\mu,a}\rangle. \tag{22}$$

So far no approximations have been made, except of course, for ignoring the switching on and switching off times in the beginning (see, however, sec IIC). The Hamiltonian H of eqn (18) can be thought of as $H_0 = H_A + H_S$ perturbed by $\frac{Q_A Q_S}{T}$. Using the fact that $\frac{Q_A Q_S}{T}$ is a small perturbation and that the eigenstates of H_0 are of the form $|\nu\rangle|a\rangle$, perturbation theory gives

$$\begin{aligned} |\Psi_{\mu,a}\rangle &= |\mu\rangle|a\rangle + O(1/T) + ... \\ E(\mu, a) &= \mu + E_A(a) + \frac{1}{T}\langle Q_S\rangle_\mu\langle Q_A\rangle_a + ... \end{aligned} \tag{23}$$

An important qualification needs to be made here. It is important for eqn(23) to hold that $|\mu\rangle|a\rangle$ be a *non-degenerate* eigenstate of $H_0 = H_A + H_S$, except when the degeneracy arises solely due to the degeneracy of the eigenstates of H_A. Otherwise, even in the limit $T \to \infty$, the exact eigenstates of H do not approach $|\mu\rangle|a\rangle$. We discuss this aspect in more detail in the following subsection with the help of two illustrative examples.

On substituting eqn(23) in eqn(22) and taking the large T limit yields

$$|t = T\rangle = \sum_b e^{i(\nu T + E_A(b)T + \langle Q_A\rangle_b\langle Q_S\rangle_\nu)}d_b|b\rangle|\nu\rangle. \tag{24}$$

We now introduce the operator

$$Y = \sum_b \langle Q_A\rangle_b|b\rangle\langle b|. \tag{25}$$

It is important to note that the operator Y is *a property of the apparatus alone and does not depend on the system.* In terms of Y, the above eqn can be recast as

$$|t = T\rangle = e^{i\nu T}e^{iH_A T}e^{iY\langle Q_S\rangle_s}|\phi\rangle|\nu\rangle. \tag{26}$$

If $|\phi\rangle$ of the apparatus is so chosen that it is peaked around a value x_0 of the operator X (the pointer variable) conjugate to Y i.e $[Y, X] = i\hbar$,

$$e^{iX\langle Q_S\rangle_\nu}|\phi(x_0)\rangle = |\phi(x_0 + \langle Q_S\rangle_\nu)\rangle. \tag{27}$$

Thus, modulo the issue of the "spreading of the pointer position" by H_A, which is present in any case even in the special case discussed earlier, the protective measurement of $\langle Q_S\rangle_\nu$ without disturbing $|\nu\rangle$ is a generic possibility. It should of course be pointed out that on the one hand it may not always be possible to physically realize the operator Y, and on the other hand an operator canonically conjugate to Y need not always exist. For example, there is no operator canonically conjugate to X^2. These and the restrictions due to degeneracy of H_0 may severely restrict the choice of realistic possibilities.

B. Degeneracy of H_0-eigenstates

As we discussed earlier, in order that eqn(23) holds, we require that $|\mu\rangle|a\rangle$ be a *non-degenerate* eigenstate of $H_0 = H_A + H_S$. However, the case where such degeneracy is due to the degeneracy of eigenstates of H_A alone, is not really a problem as a suitable basis in the degenerate subspace can be chosen in terms of which eqn(23) still holds good. We give two examples to clarify this aspect.

i) Two Harmonic Oscillators

Let us consider the situation where both the apparatus and the system are harmonic oscillators with frequency ω. Thus

$$\begin{aligned} H_A &= P^2/2M + 1/2M\omega^2 X^2, \\ H_S &= p^2/2m + 1/2m\omega^2 x^2. \end{aligned} \tag{28}$$

The energy eigenvalues for the eigenstates of this combined system labeled by $|N, n\rangle = |N\rangle|n\rangle$ are

$$E(N,n) = \hbar\omega(N+n+1). \quad (29)$$

For example, the state $|0,0\rangle$ is non-degenerate, but the states $|1,0\rangle, |0,1\rangle$ are degenerate. Now consider the adiabatic interaction

$$H_I = g(t)X \cdot x. \quad (30)$$

Let us concentrate on a degenerate subspace in the sum over (μ, a) in eqn(22). For illustration, let us choose the subspace with energy $E(0,1) = E(1,0)$. The unperturbed states are $|1\rangle|0\rangle$ and $|0\rangle|1\rangle$ respectively. The interaction H_I lifts the degeneracy and the eigenstates of $H = H_0 + H_I$ are

$$|\pm\rangle = \frac{|1,0\rangle \pm |0,1\rangle}{\sqrt{2}}, \quad (31)$$

with energy eigenvalues $E_\pm = 2\hbar\omega \pm g\lambda$ where $\lambda = \langle 0|X|1\rangle \cdot \langle 1|x|0\rangle$. Thus if the initial state were of the type $\sum_N d_N|N\rangle|0\rangle$, the contribution in eqn(22) proportional to d_1 would be

$$e^{iE_+T}\langle +|1,0\rangle|+\rangle + e^{iE_-T}\langle -|1,0\rangle|-\rangle. \quad (32)$$

After some simplifications this reduces to

$$e^{i2\hbar\omega T}\{\cos g\lambda T|1,0\rangle + i\sin g\lambda T|0,1\rangle\}, \quad (33)$$

which in the $T \to \infty$ limit reduces to

$$e^{i2\hbar\omega T}\{\cos \lambda|1,0\rangle + i\sin \lambda|0,1\rangle\}. \quad (34)$$

This introduces strong entanglement between the apparatus and system even in the adiabatic limit and consequently no protective measurement is possible.

ii) Harmonic Oscillator Coupled to spin-1/2

Let us consider a spin-1/2 particle(system) coupled to a harmonic oscillator (apparatus). The total Hamiltonian is

$$H = P^2/2M + 1/2M\omega^2X^2 + \mu B_0\sigma_z + gX\vec{\sigma}\cdot\vec{n}. \quad (35)$$

With the choice $\mu B_0 = 1/2\hbar\omega$, we see that the states $|0\rangle|+\rangle$ and $|1\rangle|-\rangle$ are degenerate. Also, the interaction Hamiltonian $H_I = gX\vec{\sigma}\cdot\vec{n}$ is not diagonal in this degenerate subspace. Again, there will be strong entanglement between the apparatus and system even in the adiabatic limit.

What one learns from these examples is that whenever the eigenstates of H_0 are degenerate in the sense mentioned above, and when the interaction Hamiltonian H_I is not diagonal in that degenerate subspace, entanglement between the apparatus and system can not be avoided even in the adiabatic limit. These two examples are cases of what could be called "accidental" degeneracy of H_0.

It is also clear that whenever either H_A or H_S has a continuous spectrum, H_0 generically has degenerate eigenstates. As an example, consider the situation where H_A has continuous spectrum a^2, and H_S the discrete spectrum $\pm\mu B_0$. Clearly the states $|a\rangle$ and $|a'\rangle$ are degenerate whenever $a'^2 = a^2 + 2\mu B_0$. It is obvious that $a'^2 \geq 2\mu B_0$. This is an example of what we call "generic" degeneracy of H_0. Protective measurement in such cases is possible only if H_I is diagonal in the degenerate subspace. In the case when $[H_A, Q_A] = 0$, H_I is indeed diagonal in the respective degenerate subspace and protective measurement is possible as we saw in section II. When $[H_A, Q_A] \neq 0$, the situation is more complex. For H_I to be diagonal in the degenerate subspace requires $\langle a|Q_A|a'\rangle = 0$ whenever $a'^2 = a^2 + 2\mu B_0$ for the example considered ($\langle a|Q_A|a'\rangle = 0$ for *all* a, a' would have meant $[H_A, Q_A] = 0$). This already precludes the prototypical Hamiltonian for Stern-Gerlach experiments:

$$H = P^2/2M + \mu B_0\sigma_z + \mu B_i X\vec{\sigma}\cdot\vec{n} \quad (36)$$

The only reason the AAV spin-1/2 example works is because of the assumption $P^2/2M \simeq 0$. We shall see this more clearly in section IIE.

C. Switching on/off of the interaction

In our treatment so far, we have ignored the possible effects of the switching on and switching off of the apparatus- system interaction. This may appear at first to question the use of the adiabatic treatment. However, it should be borne in mind that the change in the total Hamiltonian during these periods being Q_AQ_S/T is very small and the switching on and off is really a gentle process. Therefore it is intuitively clear that no violence has been committed against the adiabaticity of interactions. Nevertheless, it is desirable to put this intuitive feeling on a firmer mathematical ground to make sure nothing subtle has been missed out.

For this purpose let us assume that the interaction is smoothly switched on during the period $0 \leq t \leq \Delta T$. During this period let the function $g(t)$ be smooth and bounded by $1/T$ i.e $|g(t)| \leq 1/T$. We can also arrange for $g(t)$ to be monotonically increasing, but this is not crucial.

Now let us divide the interval $[0, \Delta T]$ into M equal parts of τ each. The initial Hamiltonian is then H_0 and the final Hamiltonian is $H_0 + Q_AQ_S/T$. During the interval labelled by m, the Hamiltonian is

$$H^{(m)} = H_0 + g_mQ_AQ_S \quad (37)$$

Let the exact eigenstates and eigenvalues of this Hamiltonian be $|\Psi^{(m)}_{\mu,a}\rangle$, $E^{(m)}_{\mu,a}$. As the Hamiltonian is *time-dependent* now, it is necessaryto use time-ordered products. The state at $t = \Delta T$ is given by

$$|\Delta T\rangle = \prod_m e^{iH_m\tau}|t=0\rangle \quad (38)$$

In a manner analogous to how we obtained eqn (22) we now get

$$|\Delta T\rangle = \sum_b d_b \sum_{\mu_1,\mu_2,..,\mu_M;a_1,a_2,..,a_M} e^{i\tau(E^{(1)}_{\mu_1,a_1}+E^{(2)}_{\mu_2,a_2}+...+E^{(M)}_{\mu_M,a_M})}|\Psi^{(M)}_{\mu_M,a_M}\rangle \langle\Psi^{(M)}_{\mu_M,a_M)}|\Psi^{(M-1)}_{\mu_{M-1},a_{M-1}}\rangle..... \langle\Psi^{(1)}_{\mu_1,a_1)}|\Psi^{(0)}_{\mu_0,a_0}\rangle..... \quad (39)$$

Because the Hamiltonians at adjacent time intervals $(i, i+1)$ differ by $(g_i - g_{i+1})Q_AQ_S$ which is again small and bounded by Q_AQ_S/T, we have

$$\langle\Psi^{(i+1)}_{\mu_{i+1},a_{i+1}}|\Psi^{(i)}_{\mu_i,a_i}\rangle = \delta_{\mu_{i+1},\mu_i}\delta_{a_{i+1},a_i} + (g_{i+1}-g_i)(\mathcal{A}+O(1/T))+.. \quad (40)$$

Here $\mathcal{A} = \langle\nu, b|Q_AQ_S|\nu, b\rangle$ and dots refer to terms higher order in $1/T$. Likewise, the energy eigenvalues satisfy

$$E^{(i)}_{\mu_i,a_i} = E_{\nu,b} + g_i\mathcal{A} \quad (41)$$

Combining these eqns and taking the limit M-large, one gets

$$|\Delta T\rangle = e^{i(\nu+E^A_b)\Delta T+\int_0^{\Delta T} dtg(t)\mathcal{A}}d_b|b\rangle|\nu\rangle \quad (42)$$

On comparing with eqn(24) it can be seen that the effect of smoothly switching on the interaction in the interval $(0, \Delta T)$ can be completely ignored. The same applies for the interval when the interaction is smoothly switched off too.

D. An Example with $[H_A, Q_A] = 0$

Let us now consider a specific example embodied by the Hamiltonian

$$H = \frac{P^2}{2M} + \mu B_0\sigma_z + g(t)\mu B_iP\vec{\sigma}\cdot\vec{n}, \quad (43)$$

where M is the mass of the particle with spin whose position acts as an apparatus, μ the magnetic moment of the particle, B_0 the homogeneous magnetic field that breaks the degeneracy of H_S, $B_iP\vec{n}$ a *momentum dependent* magnetic field that couples the apparatus and system degrees of freedom ($\vec{\sigma}$). Thus in this example $[H_A, Q_A] = 0$ while $[H_S, Q_S] \neq 0$. Further, $\nu = \pm\mu B_0$ while $E_A(a) = a^2/2M$. We take the initial state to be

$$|t=0\rangle = |\phi(\epsilon,0)\rangle|+\rangle, \quad (44)$$

where $|\phi(\epsilon,0)\rangle$ is a wave packet of width ϵ centered at $x = 0$. It is clear from the general discussion that in this case $Y = P$ and that the pointer is the center of the wave packet. In position representation

$$\langle x|\phi(\epsilon,0)\rangle = \epsilon^{-1/2}\pi^{-1/4}e^{-\frac{x^2}{2\epsilon^2}}. \quad (45)$$

We can decompose this wave packet in terms of the plane wave states (eigenstates of H_A)

$$d(a) = \frac{1}{\sqrt{2\pi}}\int dxe^{-iax}\langle x|\phi(\epsilon,0)\rangle. \quad (46)$$

One obtains

$$d(a) = \pi^{-1/4}\epsilon^{1/2}e^{-\frac{a^2\epsilon^2}{2}}. \quad (47)$$

Combining these details with eqn(13) one finds that in the case of this example

$$|t=T\rangle = e^{i\mu B_0T}e^{i\frac{P^2}{2M}T}e^{iP\mu B_i\langle\vec{\sigma}\cdot\vec{n}\rangle_+}|+\rangle|\phi(\epsilon,0)\rangle. \quad (48)$$

The operator $e^{iP\mu B_i\langle\vec{\sigma}\cdot\vec{n}\rangle_+}$ only shifts the center of the wave packet without changing its width and $e^{i\frac{P^2}{2M}T}$ only spreads the wave packet without shifting the center. Thus we find

$$|t=T\rangle = e^{iB_0T}|+\rangle|\phi(\epsilon(T), \mu B_i\langle\vec{\sigma}\cdot\vec{n}\rangle)\rangle, \quad (49)$$

where

$$\epsilon(T)^2 = \frac{1}{2}(\epsilon^2 + \frac{T^2}{M^2\epsilon^2}) \quad (50)$$

is the standard formula for the spreading of the wave packet. One may note that the spread in the pointer position in this example is independent of the system state.

E. AAV Spin-1/2 Example

The AAV example of protective measurement on a spin-1/2 state by an inhomogeneous magnetic field attracted a lot of criticism [10–17]. We present here what we think is a better way to look at this example in order to avoid any confusion. We take the inhomogeneous field to be $B_ix\vec{n}$. We take $H_A = 0$ or equivalently ignore $P^2/2M$. The relevant Hamiltonian is

$$H = -\mu B_0\vec{\sigma}\cdot\vec{\tilde{n}} - \mu g(t)B_ix\vec{\sigma}\cdot\vec{n}. \quad (51)$$

As before, $g(t)$ is taken to be $\frac{1}{T}$. *It should be noted that $B_0\vec{n}$ is an a priori unknown magnetic field.* Consequently we shall not assume anything about the size of B_0. The initial state is chosen to be

$$|t=0\rangle = e^{ip_0x}|\tilde{\mp}\rangle; \qquad \vec{\sigma}\cdot\vec{\tilde{n}}|\tilde{\pm}\rangle = \pm|\tilde{\pm}\rangle. \quad (52)$$

It should be emphasized that this initial state is a priori unknown. The Hamiltonian of eqn(37) is the Hamiltonian of the spin-1/2 particle in the effective magnetic field

$$\vec{B} = B_0\vec{\tilde{n}} + B_i\frac{x}{T}\vec{n}, \quad (53)$$

whose eigenstates are given by

$$H|\pm\rangle = \pm\mu B|\pm\rangle. \tag{54}$$

Consequently, the state at $t = T$ is given by

$$|t = T\rangle = [\cos\frac{\theta}{2}e^{i\mu BT}|+\rangle + \sin\frac{\theta}{2}e^{-i\mu BT}|-\rangle, \tag{55}$$

where θ is the angle between $\vec{B}$ and $\vec{\tilde{n}}$. As $T \to \infty$, $\theta \to 0$ and $|+\rangle \to |\tilde{+}\rangle$. Also

$$B \to B_0 + B_i\frac{x}{T}\vec{n}\cdot\vec{\tilde{n}}. \tag{56}$$

Thus

$$|t = T\rangle \to e^{i\mu B_0 T}e^{i(p_0+\mu B_i\vec{n}\cdot\vec{\tilde{n}}x)}|\tilde{\pm}\rangle. \tag{57}$$

Hence the momentum of the apparatus shifts by $\mu B_i\vec{n}\cdot\vec{\tilde{n}} = \langle\mu B_i\vec{\sigma}\cdot\vec{n}\rangle_{\tilde{\pm}}$, while the system remains in the same state to begin with.

The language used inadvertently by AAV in describing this example has, in our view, been partly responsible for some of the misunderstandings about the AAV proposal engendering a class of criticisms in [10–17]. For example AAV say, "B_0 is very large compared to the Stern-Gerlach field". This unnecessarily gives the impression that B_0 is a priori known and consequently $|\tilde{\pm}\rangle$ is a priori known too. A less confusing way to state this would have been to say that because of adiabaticity the Stern-Gerlach field $B_i\frac{x}{T}$ can be made much smaller than any B_0. Likewise, AAV state, "to see the transition from the usual Stern-Gerlach case, we may gradually increase B_0 from 0". This too gives the same false impression of B_0 being known (and hence controllable) a priori. In fact, while the usual Stern-Gerlach set up involves an impulsive transition, the modified Stern-Gerlach set up involves an adiabatic transition. This can be understood as arising out of tuning B_0 only in a formal way.

III. ASSESSING THE CRITICISMS

The proposal of protective measurements drew a lot of criticism on various counts [10–17]. Although there has been an attempt to clarify some of these misunderstandings by the original authors themselves [8], many points remain to be clarified. In this section we review the various criticisms and assess their relevance to the issue of protective measurements.

A. Are we measuring at all?

Schwinger [10] raised the following objections to the AAV proposal: i) even in the conventional Stern-Gerlach set up, as the SG-field is weakened, the two beams begin to overlap and no SG-measurement is performed, ii) repeated SG-measurements have already demonstrated the probability amplitude (epistemological) interpretation of the wave function.

Unlike the response of Aharonov and Anandan to this [18], we do agree with Schwinger that the effective SG-field is weak, because of the $\frac{1}{T}$ factor. But the circumstances are otherwise quite different from an usual SG-measurement. Since the interaction time in protective measurements is very large, even a weak SG-field is able to produce a measurable shift in the apparatus pointer position.

Regarding the second point made by Schwinger, it should be emphasized that AAV do not claim to associate reality with all wave functions. For example, the wave function for unstable systems can only be interpreted statistically. Also, repeated modified-SG (protective) measurements are indeed consistent with treating the wave function as " real".

B. Are we measuring a known state?

Rovelli [12], and, Samuel and Nityananda [17] have objected to this proposal on the ground that the fact that the wave function does not collapse is a trivial consequence of it being an eigenstate of the dominant Hamiltonian to start with. Though what they say about entanglement is correct, they overlook the crucial fact that the shift in the pointer is proportional to the expectation value of an operator which *doesn't commute* with this dominant Hamiltonian. Thus one *measures* the expectation value of an arbitrary operator of the system, while the wave function doesn't collapse for obvious reasons.

Another objection of these authors is that the wave function has to be known *a priori* in order to make a protective measurement. This claim is not completely correct, because all that is required in the analysis of protective measurements is that the system is in a non-degenerate eigenstate of its Hamiltonian, allowing for the possibility of the situation where the Hamiltonian and the state may be unknown. Indeed, one can find situations where one may know that a system is in an eigenstate without knowing the Hamiltonian. An example is a trapped atom, where the potential may not be known before hand, but one does know that after a sufficiently long time the atom is to be found in the ground state. Protective measurement, in principle, allows the measurement of any operator of the trapped particle, without destroying the state.

Alter and Yamamoto [15] have constructed an interesting example of a type of measurement whereby the system (called signal by them) and the apparatus (called the probe) maintain *exact* disentanglement after the measurement. This is achieved by using the following interesting property of coherent states of a harmonic oscillator: For a Hamiltonian $\hat{H} = \hbar\kappa(\hat{s}^\dagger\hat{p} + \hat{s}\hat{p}^\dagger)$,

$$\hat{U}(t)|\beta\rangle_s|\gamma\rangle_p = |a\beta - ib\gamma\rangle_s|a\gamma - ib\beta\rangle_p, \tag{58}$$

where $\hat{U} = e^{i\hat{H}t}$, and $\hat{s}, \hat{s}^\dagger, \hat{p}, \hat{p}^\dagger$ are the annihilation and creation operators of the system and probe respectively; further, $a = \cos\kappa t$ and $b = \sin\kappa t$. Now they take the squeezed coherent state $|\alpha, r\rangle_s$ as the system state and the squeezed vacuum state $|0, q\rangle_p$ as the probe state. The abovementioned property of coherent states then implies that the disentangled state $|\alpha, r\rangle_s|0, q\rangle_p$ remains disentangled under the unitary evolution $\hat{U}$ provided $q = -r + i\phi$ for any arbitrary phase ϕ. Their idea is then to make a measurement on the probe to infer an observable in the signal state, undo the "deterministic change" of the system by driving it back to its original state through a classical field, and repeat this process as many times as one needs. They have called this a "protective measurement" because measurements are being carried out on the system while maintaining the ability to restore the system to its original state. The price they had to pay for this was the full a priori knowledge of the system state. Hence they concluded that full a priori knowledge of the state is needed for protective measurements.

Aharonov and Vaidman [19] criticized this work on the basis that the squeezed state they use is not a non-degenerate eigenstate of the harmonic oscillator Hamiltonian, and hence does not satisfy the criterion for protective measurement. Also, [19] claim that the scheme of Alter and Yamamoto allows for disentanglement to be maintained only when certain observables are measured, much the same way as in eigenstate measurement or in "ideal von Neumann" measurements. In their rebuttal to this, Alter and Yamamoto [16] have emphasized that one can measure *all* the observables associated with the signal. They further asserted that in their scheme entanglement is *exactly* avoided while the protective measurement scheme of AAV avoids this only approximately. We fully agree with this latter remark, and shall analyze its true import a little later.

As we see it, the scheme of [15] is quite different from that of AAV and suffers from the requirement of full a priori knowledge of the state which is not a restriction on the AAV proposal. On the other hand this scheme is attractive because it avoids entanglement exactly, and is yet another candidate scheme to measure expectation values of observables in the single quantum state without irretrievably destroying it. To this extent it appears reasonable to call the scheme of [15] also a protective measurement, even if the single quantum state does not satisfy the criterion laid out by AAV.

One of the objections raised by Ghose and Home [13] (in addition to stating that protective measurements require the specification of the state) is that Aharonov *et al* have not solved the problem of wave function collapse. Protective measurement does not solve the problem of wave function collapse, and Aharonov *et al* have not claimed otherwise as they state quite explicitly in [8]. The crucial point here is that there is no entanglement between the system and the apparatus after the adiabatic interaction. So, if an actual measurement, by whatever mechanism, is made on the apparatus, which *irreversibly* registers the outcome, the wave function of the *system* will not collapse. This is similar to an eigenstate measurement using conventional method, where the wave function of the system does not change during the process of measurement, so the question of collapse, as far as the wave function of the system is concerned, doesn't arise. The wave function of the *apparatus*, on the other hand, does "collapse" in the sense that the outcome has to be registered in an irreversible way. This aspect of the measurement problem is certainly not solved by protective measurements.

C. Is the final state entangled?

The most serious attack on the idea of protective measurements can be made on the ground that in realistic situations, the wave function of the system apparatus combine is still entangled, though the degree of entanglement can be made arbitrarily small, the probability of finding the system in a state orthogonal to the initial state being of order $\frac{1}{T^2}$. This is so because in first order perturbation theory the correction to the energy eigenstate is orthogonal to it. For ensemble measurement, this small "corruption" is inconsequential as it will affect the distribution of the outcome very little. By working with suitably large ensembles one can isolate and control this admixture. This is the reason why adiabatic theorem works in the conventional interpretation of quantum mechanics. For a single system, however, even an extremely tiny entanglement can have disastrous consequence as a single measurement can yield any outcome whose probability is non-zero, resulting in a collapse to the small admixture.

The issue of entanglement has also been raised by Choudhury, Dasgupta and Datta [14] as well as Alter and Yamamoto [16]. We have, however, some objections to the technical treatment of Ref. [14]. They use small time evolution equations repeatedly in their paper, make unwarranted restrictions like simultaneous commutativity (or lack of it) of Q_A, Q_S with H_A, H_S respectively, etc. They also argue, fallaciously, that entanglement persists even in the adiabatic limit. This is a consequence of their ignoring the fact that the support for the wavefunction where this happens is exponentially small.

However, these authors stress the point that there are subtleties regarding the reading of the pointer position. In fact they correctly emphasise the point that the spread in the wavepacket of the apparatus must be handled and that the burden of protective measurements is passed on to a measurement of the pointer position. We have fully analyzed this problem in sections IV and V.

We fully concur with Alter and Yamamoto [16] regarding the serious consequences of entanglement, however small, for measurements on single systems. As a practi-

cal remedy, one could use a small number of systems prepared in identical states so that the small entanglement would not spoil each of the protective measurements performed on this small number. That, however, precludes attaching any ontological meaning to the wave function.

IV. "READING OUT" THE POINTER POSITION

A. "Spreading" of the pointer

Having established the fact that an adiabatic interaction makes it possible that the center of the wave packet of the pointer shifts by an amount proportional to the expectation value of the measured observable, we now move over to the issue of retrieving the information about the center of the wave packet. One can see that in any setup for protective measurements the pointer wave packet will spread simply because the detected pointer variable doesn't commute with the free Hamiltonian of the apparatus. Condition for adiabaticity requires that the interaction of the system with the apparatus be for as long a duration as possible. However, the increased spreading of the wave packet of the pointer would interfere with resolving the shift of the center. This aspect of protective measurements was completely overlooked in the original AAV proposal and, as we shall see in this section, it is crucial for protective measurements to work.

In order to obtain a detectable shift in the pointer position, it seems reasonable that the increase in the width of the wave packet should be at least smaller than the shift. In the example discussed in section IID, we may compare the square of the width of the wave packet $(\epsilon^2 + \frac{T^2}{M^2\epsilon^2})$ with the square of the shift in the position of the wave packet, which is $\langle Q_S\rangle_\nu$. Thus, to have a good measurement, $T < \langle Q_S\rangle_\nu \epsilon M$. From this expression one can see that in order to increase T, as one would desire for an adiabatic interaction, one can only increase the mass M of the particle. On the other hand, if the measured expectation value $\langle Q_S\rangle_\nu$ is very small, T also has to be small in order to resolve the shift in the pointer from the spread. So even in the case $[Q_A, H_A] = 0$, the spreading of the wave packet is unavoidable, and hence puts a limit on the time of the interaction, which in turn would interfere with making the interaction adiabatic.

From the analysis of the case $[Q_A, H_A] = 0$, one would recall that the initial apparatus state is a wave packet of eigenstate of the operator conjugate to Q_A. Now because $[Q_A, H_A] = 0$, that operator doesn't commute with H_A. This will lead to a spreading of the wave packet under the action of the free Hamiltonian of the apparatus H_A. In order that the wave packet doesn't spread very fast, the initial width of the wave packets should not be too small. The spread will be more as time increases, and so one should try to keep the measurement time as small as possible to avoid spreading. But in protective measurements the interaction has to be adiabatic. So, one has to strike a balance between the spreading of the wave packet and the time of interaction.

Several conceptual issues arise even though the general formalism shows a way of measuring expectation values of observables without disturbing the (single) state. What has been shown is that this protective way of measurement shifts the pointer position by an amount depending on the expectation values of observables in the state of the single system as opposed to being shifted by all possible eigenvalues of the observable in the conventional measurement picture. The implication is that the measurement of the pointer position results in a measurement of the expectation value.

B. Nature of the apparatus

This raises some fundamental issues. According to the quantum mechanical lore, no single measurement of an observable in a quantum state yields the value of the observable. Among the many critics of the AAV proposal only Choudhury, Dasgupta and Datta [14] emphasized this fundamental problem. To understand this issue properly it should be understood that the wave packet (in the example of section II) was used to model an apparatus. According to the conventional interpretation of quantum mechanics the apparatus has to be treated as being "classical". More precisely, the ideal apparatus must satisfy the following conditions:i)superposition of pointer states should not be realizable and ii)the outcome of the *measurement* of the pointer state should itself be dispersion-free. That the wave packet model for the apparatus used had associated with it the dispersion ϵ would then be interpreted as an artifact of the model. To rephrase Penrose [20], *even though the model of the apparatus has not been delicately organized in such a way that the adiabatic interaction is magnified to a classically observable event, one must consider that it could have been so organized*. Only a more satisfactory model of the apparatus would lead to a resolution of these issues. It should be stressed that the requirement of the non-realizability of the superposition of pointer states is an important prerequisite for any such model and this may necessitate a more complete analysis including agencies for decoherence as considered in [21]. If one accepts this interpretation, a single protective measurement would yield the expectation value of a chosen observable in the state of the single quantum system, which moreover, is left undisturbed by the measurement process.

The sceptic may argue that when such a consistent treatment of the apparatus is made, the conclusions of the present analysis may also not hold! Then one will have to reckon with the quantum nature of the apparatus used in the foregoing analysis, and introduce the inevitable classical apparatus at a later stage.

In that case the wave packet dispersion ϵ should be taken seriously and a number of difficulties seem to arise.

A single measurement done on the wave packet will not yield the location of the center. One possibility is that we consider adiabatic coupling of a single quantum system to an ensemble of apparatuses and make measurements on the ensemble of apparatuses to determine the pointer position. This is not such an unreasonable arrangement. For example, the ensemble of apparatuses could be a beam of atoms interacting adiabatically with the spin of the system. Such an ensemble approach inevitably carries with it uncertainty in the knowledge of the position of the apparatus. However, the pointer position which is the average of the outcome of these position measurements, can be determined with arbitrary accuracy.

C. Repeated measurement of a single state

The reason one was forced to consider an ensemble of measurements in the conventional measurement was that the (impulsive) coupling of the system to the apparatus resulted in an entangled superposition where all possible pointer positions could be realized with appropriate probabilities. In contrast, in the protective measurements only a single pointer position is chosen. This affords a more interesting alternative to considering an ensemble of apparatuses as argued above. Since the state of the system is unaltered and the expectation value of observables in the state of the single quantum system is given by the *shift* of the pointer position and not the pointer position itself, it is possible to consider the coupling between a single apparatus and the system and make repeated measurements on the (single) apparatus. Again, the reason why conventional measurements fail in this regard is that there every act of measurement irretrievably changes both the system state and the apparatus state. In the case of the protective measurements too, the state of the apparatus itself is continually being altered by the measurement in an unpredictable manner. But the shift between two successive measurements constitutes a measurement of $\langle Q_S \rangle$ and its average value can be determined by performing a large number of such measurements. In practice, the measurement of the position of the pointer can be made with a suitably small uncertainty and the subsequent measurement done after an interval not too long to increase $\epsilon(t)$ but long enough to justify the adiabaticity. Such considerations will play an important role in practical implementations.

One must however point out some caveats. Strictly speaking, even if the wave-packet is sharply peaked, the first measurement of the position can yield any value not necessarily centered around the mean value. Whether this will render useless the idea of repeated measurements on a single apparatus is to be settled by more careful examinations of the points raised. This brings us again to the point mentioned earlier that the wave-packet as a model of the apparatus must provide, if not dispersion-free measurements, that at least the measured values of the pointer position are close to its mean.

D. Quantum Nondemolition Measurement of the Apparatus

There is yet another interesting way out of the problem of measuring the shift of the wave packet of the pointer. This is based on repeated weak quantum nondemolition (QND) measurements [22] performed on the *apparatus*. Recently Alter and Yamamoto [23] analyzed the problem of a series of repeated weak QND measurement on a quantum system, to address the question of getting information about the unknown wave function of a single quantum system from such measurements. They concluded that it is possible to obtain the mean value of an observable in an unknown state, but no information can be obtained about the uncertainty of the observable. Hence one cannot obtain any information about the wave function. Also, the state is completely altered in the process.

Their scheme is best illustrated through the first of the two examples they consider in [23]. This is a series of photon number QND measurements performed on a single wave packet of light. The probe(apparatus) is a squeezed coherent state $|\alpha_0, r\rangle$ with real squeezing parameter r. The signal and probe are correlated through an unitary transformation $\hat{U} = e^{i\mu \hat{n}_s \hat{n}_p}$, where $\hat{n}_s, \hat{n}_p$ are the photon number operators for the system and probe respectively. The signal photon number is inferred from measuring the second quadrature of the probe. A series of such measurements yields $\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, ...$ for the inferred photon number of the signal. The photon number distribution in the unknown initial state is taken to be $P_0(n) = N[n, n_0, \Delta_0^2]$, with unknown n_0, Δ_0^2, where $N[x, x_0, \sigma^2] = (2\pi\sigma^2)^{-1/2} exp[-(x-x_0)^2/2\sigma^2]$ is a normalized normal distribution; here Δ_m^2 is the uncertainty due to measurements and is controllable as in classical measurements.

With each measurement, the system state *changes* and the photon number distribution of the signal after k measurements becomes $P_k(n) = N[n, n_0^{(k)}, \Delta_k^2]$, with $n_0^{(k)} = \Delta_k^2[\frac{n_0}{\Delta_0^2} + \frac{\sum \tilde{n}_i}{\Delta_m^2}]$ and $\Delta_k^2 = (\frac{1}{\Delta_0^2} + \frac{k}{\Delta_m^2})^{-1}$. The important features of this example to concentrate on are: i) $P_k(n_0^{(k)})$, the diffusion of the centre after k measurements, is given by $N[n_0^{(k)}, n_0, (k/\Delta_m^2)\Delta_0^2\Delta_k^2]$. *This distribution is centred at* n_0. ii)If $\bar{n} = \sum_{i=1}^{k} \tilde{n}_i/k$ and $\bar{\Delta n^2} = \sum_1^k (\tilde{n}_i - \bar{n})^2/(k-1)$ are the mean and variance of the outcome of measurements $\tilde{n}_1, \tilde{n}_2, ...$, the probability distribution of $\bar{n}$ and $S = [(k-1)/\Delta_m^2]\bar{\Delta n^2}$ are given by $N[\bar{n}, n_0, \Delta_0^2 + \Delta_m^2/k]$ and $\chi^2[S, (k-1)]$ respectively, where $\chi^2[x, \nu]$ is the chi-squared distribution of the variable x which is centred at ν. *Thus, while* $\bar{n}$ *is a "good" estimator for* n_0, $\bar{\Delta n^2}$ *being centered at* Δ_m^2 *has nothing to do with the initial uncertainty* Δ_0^2. iii) Eventually,

the width of the distribution $P_k(n)$ becomes zero which means the signal becomes an eigenstate of photon number with eigenvalue n_0.

While the conclusions of [23] were negative as far as using repeated weak QND measurements to determine the unknown wave function of a single system, it appears tailor-made to solve the problem of "reading the pointer position" in protective measurements. Thus we apply their scheme not to the system part of the protective measurement set ups, but to the apparatus part instead. Then we can get information about the center of the wave packet, which in the protective measurement scheme carries information on the expectation values of observables in the system state, through repeated measurement of the (quantum) apparatus. There is also the added advantage that the variance in the outcome of these repeated measurements has nothing to do with the spread in the wave packet of the apparatus. The uncertainty in the measured values of $\langle Q_S \rangle$ will therefore be more like errors in classical measurements which are controllable. Hence there need not be any uncontrolled uncertainty in the reconstruction of the original state. The concerns expressed in section IVA get mitigated in an elegant manner.

This example comes closest to realizing the ideals of a classical apparatus but nevertheless dealing with an apparatus that is treated quantum mechanically. We need not care about the fact that it doesn't give us information about the variance, as all we need to know, in order to complete the protective measurement, is the position of the center of a pointer wave packet. There is an added bonus to this method in the sense that n_0 can be determined as the average of the outcome of sequence of measurements *as well as by performing an eigenstate measurement on the eventual apparatus state!*. We also don't care that the original state is destroyed after the measurement, because for us it is the state of the apparatus that is destroyed, and not of the system.

Thus one may proceed with a protective measurement by first allowing an adiabatic interaction of the system with an apparatus which can be treated quantum mechanically. This would result in a shifted wave packet of the pointer. One can then do a series of *weak* QND measurements on this wave packet to get the position of the center. This seems the most promising possibility for experimentally realizing protective measurements.

V. SOME FINAL REMARKS

A. Restrictions on the apparatus

From our general discussion of protective measurements, it is clear that many restrictions may have to be imposed on the kind of apparatus to be used. By an apparatus in this context, we shall mean a specification of H_A, Q_A. In the general case where $[Q_A, H_A] \neq 0$, it is not clear whether the operators X and Y can be physically realized in an actual set up. Also, as already pointed out, it may not always be possible to even find a X that is canonically conjugate to Y.

The other important restriction on the apparatus comes from the requirement that $|\nu\rangle|a\rangle$ in our general treatment should not be a degenerate eigenstate of $H_0 = H_S + H_A$ unless the perturbation gQ_AQ_S is *diagonal* in the degenerate subspace. Generically, H_A should not have a continuous spectrum, though in a specific example given in IID, continuity of the spectrum was not a problem because the perturbation there was diagonal in the degenerate subspace. In fact, in all cases where $[H_A, Q_A] = 0$, the perturbation will be diagonal in the degenerate subspace (recall that $|\nu\rangle$ is a *non-degenerate* eigenstate of H_S). These considerations rule out, for example, the prototypical Hamiltonian in discussions of the Stern-Gerlach set up i.e $H = P^2/2M + \mu B_0\sigma_z + \mu B_i X \vec{\sigma}\cdot\vec{n}$. As emphasized before, AAV in their spin-1/2 example chose $H_A = 0$. But once this is relaxed, the difficulties stressed here become relevant.

The other important point to emphasize is that $\frac{Q_AQ_S}{T}$ should be a well-defined perturbation over H_0 in the sense that its matrix elements in the basis spanned by the eigenstates of the unperturbed Hamiltonian should exist. This too rules out the prototypical Hamiltonian in the discussions of the Stern-Gerlach model mentioned above, because expectation value of x in any plane-wave state does not exist. Not only should the matrix elements exist, at least some of the diagonal matrix elements of Q_A should be nonvanishing, as otherwise there will be no shift in the pointer position. This, for example, rules out a linear position coupling in the case of a Harmonic oscillator.

One might have have thought that the Stern-Gerlach Hamiltonian could have been used with some sort of "regularization" like putting the particle in a box, or treat the free particle as a harmonic oscillator with a very tiny ω. But both these are unsatisfactory for the purpose of protective measurements because in the first case $\langle |x| \rangle$ in the eigenstates of $P^2/2M$ with box-boundary conditions is always the same and is at the center of the box. Then the operator Y of our general treatment is the identity operator for which there is no canonical conjugate. Physically, this means that the adiabatic interaction only produces an overall phase which is of no consequence in shifting the pointer position. The second alternative of treating the free particle as the limit of a harmonic oscillator with vanishing frequency is also no good as in this case the expectation value of x in the oscillator energy eigenstates vanishes and there will be no pointer position shift. In these cases a more rigorous handling than what perturbation theory offers may be needed.

It is not clear that even the case where $[Q_A, H_A] = 0$ is easily realizable experimentally. In the example of IID one needs a momentum dependent magnetic field. While it is always possible to experimentally create a position dependent magnetic field by using inhomogeneous fields,

it is not clear how one would create the former.

These restrictions on Q_A, Q_S are not warranted in the conventional i.e impulsive measurements as there $Q_A Q_S$ is dominant and $\int H_A, \int H_S$ can be neglected in comparison (the integration is over the duration of the impulse).

B. Does it *really* work for a single system?

In the entire analysis it has been assumed that entanglement effects (between the apparatus and the system) can be made arbitrarily small as T is made large. In the case of conventional measurements, a small contamination of the wave function will also have only a small statistical effect. With a large enough ensemble of states, the effect of such small admixtures in the wave function can be controlled. In the case of protective measurements the situation is radically different. However small the amplitude for entanglement in the large T limit, the outcome of the first measurement on the single system can always be states of the system and apparatus which are part of the small amplitude. This would have a deleterious effect on the subsequent measurements. It is clear that this potential problem persists no matter how large T(or how small $\frac{1}{T}$) is made. Stated differently, however large T is made, the *possibility* that even a "protective measurement" projects the system into a state orthogonal to its initial state can never be ruled out. The fact that the *calculated probability* for this to happen could be extraordinarily tiny is of no consequence because for a single system under such circumstances, probabilistic concepts are inapplicable. To illustrate this in the specific context of the example of sec IIE, the angle θ is always non-zero though very small and the original spin is precessing around the unknown magnetic field with this inclination. Quantum mechanically speaking, any measurement can realise both the initial state $|\tilde{+}>$ as well as its orthogonal complement $|\tilde{-}>$. This may well be the most formidable obstacle to realizing protective measurements with certainty. *In this sense it is the conventional interpretation of the wave function and measurements that is protected against the vagaries of statistical fluctuations*

C. Philosophical issues

The idea of a protective measurement, like its conventional counterpart, also has some philosophical issues associated with it. Because of the fact that there exists a possibility of measuring the expectation value of an observable from an unknown wave function when it is an energy eigenstate, one might tend to associate a reality with energy eigenstates. If one believes what can be measured is "real", then the energy eigenstates appear, on first glance, to satisfy this condition, and seem to be special in this regard. On the other hand, the ubiquitous tiny entanglement makes it impossible to make a measurement on one single system, with complete certainty, as we discussed before. If the entanglement is really tiny, it may not be that bad from a practical point of view, in the sense that a small number of such measurements are likely to give the right answer. But it still precludes associating a "reality" with the wave function of a *single* system.

Unruh has raised an objection to associating a "reality" with the wave function even after assuming the validity of the idea of protective measurements on a single system [11]. He argues that the energy eigenstates may be considered to have a "reality", but that cannot be concluded about any arbitrary state. For the reasons mentioned above, we do not believe one should associate a "reality" even with the energy eigenstates of a system.

Unruh also points out that "protection" in the sense used by AAV is an attribute which a system either already has or doesn't have, which means that only if a system is already in a non-degenerate energy eigenstate, can a protective measurement be performed on it. One cannot "protect" a given unknown wave function.

VI. SUMMARY

In summary, we have critically examined the idea of protective measurement of a quantum state. We have shown that the idea can be generalized to the case where the interaction Hamiltonian does not commute with the free Hamiltonian. We have also looked at earlier criticisms of the idea and conclude that most of them are not relevant to the original proposal. The relevant criticisms, we believe, are the comments by Alter and Yamamoto [16] on the omnipresent infinitesimal entanglement, comments by Choudhury, Dasgupta and Datta [14] pointing out the subtleties in reading out the pointer, and the comments by Unruh [11] on the interpretation of protective measurements. We discuss various conceptual issues involved in the process of protective measurements and infer that there are several constraints imposed on the measuring apparatus. It is pointed out that a single measurement does not yield any information. We propose two schemes as a way out of this problem. One of this involves performing repeated measurements on the single quantum system, making use of the fact that the system wave function doesn't change. The other proposal involves performing a series of quantum nondemolition measurements on the *apparatus*, which is to be treated quantum mechanically, after a single protective measurement on the quantum system. After analyzing all the issues involved, we conclude that although experimentally realizing protective measurements is a possibility, one can never perform a protective measurement on a single quantum system with absolute certainty because of the tiny unavoidable entanglement which is always there. This is sufficient ground for precluding the "reality" of

the wave function. In this sense we agree with Unruh that what the AAV proposal has achieved is a fresh understanding of the nature of measurements in quantum mechanics, rather than elevate the wave function to a new status.

On the practical side, it appears that protective measurements (where possible) can be used to determine the wave function using considerably smaller ensembles than in traditional measurements, with the added bonus that the ensemble is practically left in tact after the measurements. We give the following semi-quantitative argument in support of this. Of course, a more detailed model-specific analysis would be required to make these arguments more concrete.

Let us compare the measurement of some quantity X in both conventional ensemble measurements as well as in protective measurements. In the latter case, let us consider doing it with an ensemble of N_p identically prepared states and let the former be done with an ensemble of N_c identically prepared states. In the case of protective measurements we obtain with probability $1 - c^2/T^2$, the exact expectation value $\langle X \rangle_{exact}$ and with probability c^2/T^2 the expectation value $\langle X \rangle_\perp$ where c depends on the details of the system and $\perp$ refers to the normalised state in the subspace normal to the initial state as picked out by first order perturbation theory. It is worth noting that the relative probability $\sim 1/T^2$ as in first order perturbation the change in the wavefunction is *orthogonal* to it. This works to a tremendous advantage for realising protective measurements. Thus the error in the estimation of the expectation value by protective measurements is $c^2 \langle X \rangle_\perp / T^2$. Of course the statistical error $1/\sqrt{N_p}$ weighed with the relevant probability should also be taken into account. Combining the errors in quadrature, one gets for the estimate of error in protective measurements

$$\epsilon_p = \frac{c^2}{T^2} \sqrt{[\langle X \rangle_\perp^2 + \frac{1}{N_p}]} \tag{59}$$

The size of the conventional ensemble N_c required to match this precision is roughly $1/\epsilon_p^2$ and is given by

$$N_c = \frac{T^4}{c^4} \frac{1}{[\langle X \rangle_\perp^2 + 1/N_p]} \tag{60}$$

Thus with large enough T one can achieve substantial reduction in the ensembles required for protective measurements, for any given degree of precision in measurements.The estimate provided above is crucially dependent on ones ability to carry out the QND measurement on the apparatus as detailed in sec IVD.

This is indeed a very attractive practical spin-off for the AAV proposal. The other attractive feature, as already mentioned earlier, is that the original pure ensemble remains pure with probability $1 - c^2/T^2$ whereas in conventional measurements the original pure ensemble is *completely* destroyed in the sense that it is reduced to a mixed ensemble from which it can not be reconstructed.

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